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Irreducible maps and bilinear forms

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Abstract

Bautista showed in 1982 that the possible multiplicities of indecomposable summands of the domains and ranges of irreducible maps between modules over artin algebras are given by numerical invariants of certain bilinear forms associated with the algebra. We obtain further information about these multiplicities by relating the forms to those studied elsewhere in algebra and geometry. One spectacular result is that the allowable multiplicities for some algebras over the field of real numbers depend on J.F. Adams' determination of the number of linearly independent vector fields on a sphere.

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1. Introduction

Let D_1 and D_2 be finite-dimensional division algebras over a common central subfield k, and let $W = {}_{D_1}W_{D_2}$ be a bimodule of finite k-dimension on which k acts centrally. We say that a bilinear form over W,

$$\Phi : D_1^{m_1} \times D_2^{m_2} \longrightarrow W,$$

is *definite* if

$$\Phi(u, v) = 0 \quad \text{implies either } u = 0 \text{ or } v = 0. \tag{1}$$

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In this case the pair (m_1, m_2) will be called an *allowable dimension type* for the *quartet* (k, D_1, D_2, W).

Bautista [3] has shown that allowable dimension types encode the possible multiplicities of indecomposable summands of the domain and range of an irreducible map between modules over an artin algebra. In Section 2, we summarise his work and show that any allowable dimension type for any quartet can be obtained from this encoding.

Section 3 contains results about possible dimension types for quartets in general. One striking feature is their basefield dependence. Consider, for example, the problem of finding the allowable dimension types (m_1, m_2) for the quartet (k, k, k, k^n) for a given field k. This is equivalent to the classical problem of finding the possible number m_1 of $n \times m_2$ matrices over k with the property that *every* non-trivial k-linear combination of them has rank m_2. For k algebraically closed, $m_1 + m_2 \leqslant n + 1$ is the condition; for a field k which has an algebraic extension of degree n, it is that $m_i \leqslant n$ for $i = 1, 2$. If $k = \mathbb{R}$ and $m_2 = n$, the maximum value of m_1 is given by the Hurwitz–Radon formula, namely, if $n = 2^{4p+q}r$, with $0 \leqslant q \leqslant 3$ and r odd, then $m_1 \leqslant \mathrm{hr}(n) := 8p + 2^q$. A form which achieves the limit $\mathrm{hr}(n)$ was constructed, using algebraic methods, by Hurwitz and Radon. However, the proof that $\mathrm{hr}(n)$ is the maximum value of m_1 seems to require Adams' topological solution of the problem of vector fields on spheres [1].

Hurwitz and Radon were interested in the composition of quadratic forms, the subject of Shapiro's recent book [15]. This book also covers a wide range of related topics; in particular it contains a comprehensive treatment of the classical problem mentioned above.

In Section 4 we discuss briefly the quartet $(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}^n)$ and the fascinating interplay between algebra and topology which is part of its history. For a thorough discussion and an up-to-date treatment, see [15].

2. Irreducible maps

Let Λ be a connected artin algebra over a commutative artin ring R, local since Λ is connected. Bautista describes the structure of irreducible maps between finitely generated Λ-modules. He first shows that a map is irreducible if and only if it is the direct sum of an isomorphism and an irreducible map in $\operatorname{rad} \operatorname{mod} \Lambda$. Next he considers a map $f = (f_{ij}) : \oplus X_i^{p_i} \to \oplus Y_j^{q_j}$ in $\operatorname{rad} \operatorname{mod} \Lambda$, with $\{X_i\}$ and $\{Y_j\}$ sets of pairwise non-isomorphic indecomposable modules, and shows that f is irreducible if and only if each f_{ij} is irreducible. Thus the problem is that of describing irreducible maps between powers of indecomposable modules and this may be analysed as follows.

Suppose that M_1 and M_2 are indecomposable Λ-modules which admit at least one irreducible map from M_1 to M_2, so that $M_1 \ncong M_2$. We define a quartet (k, D_1, D_2, W) as follows: $k = R/\operatorname{rad} R$; $D_i = \operatorname{End} M_i / \operatorname{rad} \operatorname{End} M_i$, $i = 1, 2$; and $W =$

$W(M_1, M_2) = \operatorname{rad}(M_1, M_2)/\operatorname{rad}^2(M_1, M_2)$ is the 'bimodule of irreducible maps'. Any map $f : M_1^{m_1} \to M_2^{m_2}$ has all its components in $\operatorname{rad}(M_1, M_2)$ and so induces a bilinear form over W

$$\Phi : D_1^{m_1} \times D_2^{m_2} \longrightarrow {}_{D_1}W_{D_2}, \quad (u, v) \mapsto \sum u_i \phi_{ij} v_j,$$

where $\phi_{ij} \in W = W(M_1, M_2)$ is the image of the (i, j) component of *f*.

Proposition 1 [3]. *The map f is irreducible if and only if its form Φ is definite.*

In particular, if *f* is irreducible, then (m_1, m_2) is an allowable dimension type for the quartet (k, D_1, D_2, W).

The following proposition shows that any definite form over any quartet has such a realisation.

Proposition 2. *Given a quartet (k, D_1, D_2, W) and a definite bilinear form $\Phi : D_1^{m_1} \times D_2^{m_2} \to {}_{D_1}W_{D_2}$, there is an algebra Λ and an irreducible map which realises them.*

Proof. Let

$$\Lambda = \begin{pmatrix} D_1 & W \\ 0 & D_2 \end{pmatrix}, \quad M_1 = \Lambda e_{11} \quad \text{and} \quad M_2 = \Lambda e_{22},$$

where e_{11} and e_{22} are the usual idempotent matrices. Clearly, $D_i = \operatorname{End} M_i$, $i = 1, 2$. Each non-zero element $w \in W$ determines an irreducible map w_R from M_1 to M_2 which is right multiplication by $\left(\begin{smallmatrix} 0 & w \\ 0 & 0 \end{smallmatrix}\right)$ and the map $w \mapsto w_R$ induces an isomorphism from W to $W(M_1, M_2)$. Let $f : M_1^{m_1} \to M_2^{m_2}$ be the map with (i, j) component the irreducible map $w_{ij,R}$, where w_{ij} is the (i, j) component of Φ. By Bautista's proposition, *f* is irreducible. $\square$

Remark. The quartets associated with a given artin algebra Λ share some common features. Let (k, D_1, D_2, W) be the quartet associated with indecomposable modules M_1 and M_2 with $W = W(M_1, M_2) \neq 0$. Since M_1 and M_2 are finitely generated as R-modules, D_1 and D_2 are of finite dimension over the central subfield $k = R/\operatorname{rad} R$, as also is W because $\operatorname{Hom}_\Lambda(M_1, M_2)$ is a finitely generated R-module. There are also restrictions on the dimensions $l_1 = \dim_{D_1} W$ and $l_2 = \dim W_{D_2}$ consequent on l_1 being the multiplicity of M_1 as a direct summand of the domain of the sink map for M_2 and l_2 being the multiplicity of M_2 as a direct summand of the range of the source map for M_1; see [12, Section 2.3] and [13, Section 2.2], or [2, Chapter VII, Proposition 1.3], where sink maps are called minimal right almost split morphisms and source maps are called minimal left almost split morphisms. One such restriction is that $l_1 l_2$ is bounded by a constant dependant only on Λ. The proof uses the bound, given in [12], on the ratio of the lengths of the modules at the ends of an almost split sequence. For an algebra Λ of finite representation type, $l_1 l_2 \leqslant 3$ is a consequence of results in [4].

3. Definite bilinear forms

In this section, we discuss conditions for a pair of natural numbers to be an allowable dimension type for a given quartet $\mathcal{Q} = (k, D_1, D_2, W)$. As above we write $l_1 = \dim_{D_1} W$ and $l_2 = \dim W_{D_2}$. The statements in the following proposition are direct consequences of the definitions.

Proposition 3

(a) *If (m_1, m_2) is allowable for $\mathcal{Q}$, so also is any pair (n_1, n_2) with $n_1 \leqslant m_1$ and $n_2 \leqslant m_2$.*
(b) *If (m_1, m_2) is allowable for $\mathcal{Q}$, then $m_1 \leqslant l_1$ and $m_2 \leqslant l_2$.*
(c) *The dimension types $(l_1, 1)$ and $(1, l_2)$ are allowable for $\mathcal{Q}$.*
(d) *If (m_1, m_2) is allowable for (k, D_1, D_2, W') and W' is a sub-bimodule of a bimodule W, then it is allowable for (k, D_1, D_2, W).*

Our proof of the next proposition generalises the proof of a special case given by Hopf in [7]. Hopf's construction is referred to as a Cauchy product form in [15].

Proposition 4. *Let (n_1, n_2) be an allowable dimension type for each of the n quartets (k, D_1, D_2, W_r), $r = 1, 2, \ldots, n$. Let m_1 and m_2 be positive integers with $m_1 + m_2 \leqslant n + 1$. Then $(m_1 n_1, m_2 n_2)$ is an allowable dimension type for any quartet (k, D_1, D_2, W) such that W contains a copy of $W_1 \oplus W_2 \oplus \cdots \oplus W_n$.*

Proof. Given a bilinear form

$$\Phi_r : D_1^{n_1} \times D_2^{n_2} \longrightarrow W_r$$

for each of the n given quartets, there is a bilinear map

$$\Phi : D_1^{m_1 n_1} \times D_2^{m_2 n_2} \longrightarrow W' = \bigoplus_{r=1}^{m_1+m_2-1} W_r$$

defined by taking the projection of $\Phi((u_1, \ldots, u_{m_1}), (v_1, \ldots, v_{m_2}))$ into the summand W_r of W' to be $\sum_{i+j=r+1} \Phi_r(u_i, v_j)$ for all $u_1, \ldots, u_{m_1} \in D_1^{n_1}$ and all $v_1, \ldots, v_{m_2} \in D_2^{n_2}$. An easy induction on $m_1 + m_2$ shows that Φ is definite if and only if each of $\Phi_1, \Phi_2, \ldots, \Phi_{m_1+m_2-1}$ is definite. The proposition then follows from Proposition 3(d) above. □

Let $d_i = \dim_k D_i$, $i = 1, 2$. For the quartet $(k, D_1, D_2, D_1 \otimes_k D_2)$, the pairs $(1, d_1)$ and $(d_2, 1)$ are both allowable by Proposition 3(c) and so the last proposition has the following corollary.

Corollary 5. *Let (k, D_1, D_2, W) be a quartet in which W contains a direct sum of n copies of the bimodule $D_1 \otimes_k D_2$. If m_1 and m_2 satisfy $m_1 + m_2 \leqslant n + 1$, then $(m_1, m_2 d_1)$ and $(m_1 d_2, m_2)$ are allowable dimension types.*

In the next corollary, the term *twisted bimodule* means a D, D-bimodule of the form Dw with D acting on the right of w through an automorphism α of D which fixes k elementwise. For any such twisted bimodule, the map $D \times D \to Dw$, $(u, v) \mapsto u(v\alpha)w$ is a definite bilinear form and so $(1, 1)$ is an allowable dimension type for the quartet (k, D, D, Dw).

Corollary 6. *Let (k, D, D, W) be a quartet in which W contains the direct sum of n twisted bimodules. If m_1 and m_2 satisfy $m_1 + m_2 \leqslant n + 1$, then (m_1, m_2) is an allowable dimension type.*

The final results in this section concern quartets of the form (k, k, k, k^n) and are given in Chapter 14 of [15], with proofs which are minor variants of ours.

For an algebraically closed field k, any quartet has the form (k, k, k, W) with $W \simeq k^n$ and the following proposition then gives precise conditions for (m_1, m_2) to be allowable.

Proposition 7. *Let k be an algebraically closed field. Then (m_1, m_2) is an allowable dimension type for the quartet (k, k, k, k^n) if and only if $m_1 + m_2 \leqslant n + 1$.*

Proof. Corollary 6 includes the 'if' part of the proposition.

For the 'only if' part, suppose that $\Phi : k^{m_1} \times k^{m_2} \to k^n$ is a definite bilinear form. Consider the Segré embedding $\mathbb{P}^{m_1-1} \times \mathbb{P}^{m_2-1} \to \mathbb{P}^{m_1 m_2-1}$, $((u_i), (v_j)) \mapsto (u_i v_j)$, where $\mathbb{P}^m$ denotes m-dimensional projective space. The equation $\Phi(u, v) = 0$ is equivalent to n homogeneous linear equations for the $u_i v_j$. Since k is algebraically closed, the dimension of the set of solutions for these equations is at least $m_1 + m_2 - 2 - n$ (see [14, Chapter I, Section 6.1, Example 4, and Section 6.2, Corollary 5], for example). Since Φ is definite, this solution set is empty, and so $m_1 + m_2 - 2 - n < 0$ as required. □

For the quartet (k, k, k, k^n) with k not algebraically closed, the next result shows that the condition $m_1 + m_2 \leqslant n + 1$ can often be relaxed.

Proposition 8. *Suppose that there is a division algebra (not necessarily associative) of dimension d over k, with k central. Then (d, d) is allowable for the quartet (k, k, k, k^d).*

Proof. Take $\Phi : k^d \times k^d \to k^d$ to be the multiplication map for the presumed division algebra structure on k^d. $\square$

Corollary 9. *Let k be a field over which there is at least one irreducible polynomial of each degree. Then (m_1, m_2) is an allowable type for the quartet (k, k, k, W) if and only if $m_i \leqslant \dim_k W$ for $i = 1, 2$.*

4. The quartet $(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}^n)$

This section contains what we know about allowable dimension types for the quartet $(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}^n)$ as well as a little of the history of definite bilinear forms for it.

The term 'definite' was used by Hopf in his paper [7] in which he reformulated some topological ideas of Stiefel from [16] and applied them to algebra. He defined a continuous map $\Phi : \mathbb{S}^{m_1-1} \times \mathbb{S}^{m_2-1} \to \mathbb{R}^n$, where $\mathbb{S}^m$ denotes the m-sphere, to be 'odd' if $\Phi(-u, v) = -\Phi(u, v) = \Phi(u, -v)$, and to be 'definite' if, in addition, the equation $\Phi(u, v) = 0$ has no solution. Thus a definite bilinear form, in our sense, from $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ to $\mathbb{R}^n$ restricts to a definite map in Hopf's sense. The main theorem in [7] is that, if such a definite form Φ exists, then the binomial coefficients

$$\binom{n}{r} \quad \text{for } n - m_1 < r < m_2$$

are necessarily even. The corresponding result in [16] is that these conditions are necessary for the existence of $m_1 - 1$ continuous vector fields in the real projective space $\mathbb{P}^{n-1}$, which are linearly independent at each point of a subspace $\mathbb{P}^{m_2-1}$ of $\mathbb{P}^{n-1}$. The proofs of both Hopf and Stiefel are topological.

At Hopf's suggestion, Behrend [5] found a purely algebraic proof in the case where Φ is given by homogeneous polynomial functions. He used van der Waerden's then new algebraic geometric methods and noted that $\mathbb{R}$ could be replaced by an arbitrary real closed field. He also noted that the Hopf–Stiefel result can easily be recovered from the statement that, if there is a definite form from $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ to $\mathbb{R}^{m_1+m_2-2}$, then

$$\binom{m_1 + m_2 - 2}{m_1 - 1} \quad \text{is even.}$$

A proof of Behrend's result may be obtained as a special case of Theorem 2 in Section 2.2 of Chapter IV: 'Intersection Numbers' in [14]. These conditions on binomial coefficients give what seem to be the only general result on allowable dimension types for a quartet $\mathcal{Q} = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}^n)$. For more detailed results, see [15].

Dimension types of the form (m, n) for the quartet $\mathcal{Q}$ are known: the maximum value of m is given by the Hurwitz–Radon formula $\mathrm{hr}(n)$ defined in Section 1. In [7,8] and [11] it is shown that $\mathrm{hr}(n)$ is the maximum dimension of an $\mathbb{R}$-linear space of $n \times n$ matrices each of which is a scalar multiple of an orthogonal matrix; the existence of such a space is equivalent to that of a space with a basis consisting of the identity matrix and $\mathrm{hr}(n) - 1$ skew-symmetric orthogonal matrices which anticommute with one another. (See Eckmann's paper, [6], for a beautiful proof of the Hurwitz–Radon result using group representation theory.) This result shows that $(\mathrm{hr}(n), n)$ is an allowable dimension type for $\mathscr{Q}$. However, it leaves open the possibility of there being an allowable type (m, n) with $m > \mathrm{hr}(n)$, that is, of there being, for such an m, an m-dimensional linear space of $n \times n$ matrices with all but the zero matrix invertible. It seems that the only way of excluding this possibility is by noting that it would imply the existence of $m - 1$ linearly independent continuous vector fields on $\mathbb{S}^{n-1}$. This would contradict Adams' theorem, in [1], that $\mathbb{S}^{n-1}$ admits at most $\mathrm{hr}(n) - 1$ linearly independent tangent vector fields.

For up-to-date accounts of the construction of $\mathrm{hr}(n) - 1$ linearly independent vector fields on $\mathbb{S}^{n-1}$, using the representation theory of Clifford algebras see, for example, [9], [10] or [15].

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